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ONE SOLUTION OF AN AXISYMMETRIC PROBLEM OF THE ELASTICITY THEORY FOR A TRANSVERSELY ISOTROPIC MATERIAL

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A numerical-analytical method based on approximation of the sought solution by a system of basis functions is proposed to solve the boundary-value problem of axisymmetric deformation of articles made of a transversely isotropic material. An algorithm for constructing polynomial functions on the basis of invariant-group solutions is described.

Key words: *composite material, mathematical modeling, stress function, stress–strain state, axisymmetric deformation, basis functions.*

The use of composite materials in design of load-bearing structures for various purposes requires the development of numerical models and methods that take into account the specific features of the structure and behavior of such materials [1]. The stress–strain state of articles made of composite materials is estimated by conventional methods with the use of exact mathematical models based on mechanics of continuous media. Application of these methods allows one to avoid various errors caused by assumptions used in approximate calculation techniques. Simultaneously, mathematical models and algorithms should allow user-friendly computer applications. The present paper deals with transversely isotropic homogeneous materials with a plane of isotropy at each point (Fig. 1). Elements made of unidirectional composite materials (tubes, rods, sections, shells, etc.), including bodies of revolution (Fig. 2), are often used in modern flying vehicles.

According to the algorithm of solving a three-dimensional problem of the elasticity theory for a transversely isotropic medium [2], the solution of the system of second-order equations in displacements is reduced to the solution of differential equations in partial derivatives of the sixth order with respect to the displacement components u and v and of the fifth order with respect to the displacement component w in direction 1 (Fig. 1), which is perpendicular to the plane of isotropy. These equations were solved in [3–5] by a numerical-analytical approach based on approximation of the sought solution by a system of polynomial basis functions. The drawback of this approach is additional calculations caused by the necessity of identical satisfaction of the initial system of differential equations in displacements owing to an appropriate choice of internal parameters at basis functions. This drawback is eliminated in the approach proposed in the present work.

Let us consider a body made of a transversely isotropic homogeneous elastic material (domain Ω), which is bounded by one or several coaxial surfaces of revolution and is in equilibrium under the action of external forces (the volume forces are neglected). We assume that the planes of isotropy passing through each point of the body are perpendicular to its geometric axis (axis of revolution), and the force distribution is symmetric about this axis. By virtue of symmetry of the distributions of forces and elastic properties, the radial sections remain plane, and the body in the deformed state is still a body of revolution, i.e., $u_r = u_r(r, z)$ and $u_\theta = 0$, $w = w(r, z)$. The strain components are independent of θ and are calculated by the formulas [6, 7]

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial w}{\partial r}, \quad \gamma_{r\theta} = \gamma_{\theta z} = 0. \tag{1}$$

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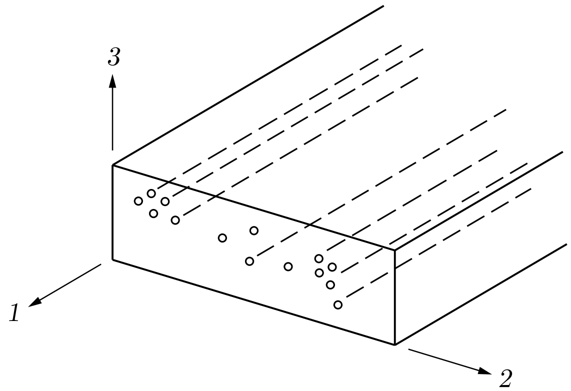

Fig. 1

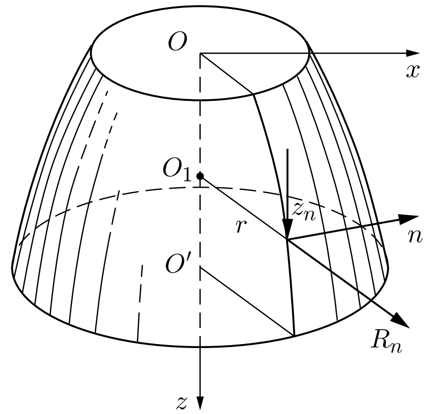

Fig. 2

Fig. 1. Model of a transversely isotropic unidirectional composite: the directions of the principal axes are indicated by 1–3.

Fig. 2. Model of a transversely isotropic body of revolution.

In this case, the relations between the stress tensor components and strain tensor components have the form

$$\varepsilon_r = a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z, \qquad \varepsilon_\theta = a_{12}\sigma_r + a_{11}\sigma_\theta + a_{13}\sigma_z,$$
$$\varepsilon_z = a_{13}(\sigma_r + \sigma_\theta) + a_{33}\sigma_z, \qquad \varepsilon_{rz} = a_{44}\tau_{rz}, \tag{2}$$

where a_{ij} are the elastic constant of the material, $a_{11} = 1/E$, $a_{12} = -\nu/E$, $a_{33} = 1/E'$, $a_{13} = -\nu'/E'$, $a_{44} = 1/G'$, $2(a_{11} - a_{12}) = [2(1+\nu)]/E = 1/G$, E and E' are Young's moduli of tension and compression in the plane of isotropy and in the direction perpendicular to this plane, ν is Poisson's ratio characterizing material constriction in the plane of isotropy under tension in this plane, ν' is Poisson's ratio characterizing material constriction in the direction normal to the plane of isotropy under tension in this plane, and $G = E/[2(1+\nu)]$ and G' are the shear moduli for the planes of isotropy and for the planes perpendicular to them (radial planes).

As was noted in [6], the stresses in the case of axisymmetric deformation of a transversely isotropic body can be expressed via one stress function

$$\sigma_r = -\frac{\partial}{\partial z}\Big(\frac{\partial^2\varphi}{\partial r^2} + \frac{b}{r}\,\frac{\partial\varphi}{\partial r} + a\,\frac{\partial^2\varphi}{\partial z^2}\Big), \qquad \sigma_\theta = -\frac{\partial}{\partial z}\Big(b\,\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\varphi}{\partial r} + a\,\frac{\partial^2\varphi}{\partial z^2}\Big),$$
$$\sigma_z = -\frac{\partial}{\partial z}\Big(c\,\frac{\partial^2\varphi}{\partial r^2} + \frac{c}{r}\,\frac{\partial\varphi}{\partial r} + d\,\frac{\partial^2\varphi}{\partial z^2}\Big), \qquad \tau_{rz} = \frac{\partial}{\partial z}\Big(\frac{\partial^2\varphi}{\partial r^2} + \frac{c}{r}\,\frac{\partial\varphi}{\partial r} + a\,\frac{\partial^2\varphi}{\partial z^2}\Big), \tag{3}$$
$$\tau_{r\theta} = \tau_{z\theta} = 0,$$

where

$$a = \frac{a_{13}(a_{11} - a_{12})}{a_{11}a_{33} - a_{13}^2}, \qquad b = \frac{a_{13}(a_{13} + a_{44}) - a_{12}a_{33}}{a_{11}a_{33} - a_{13}^2},$$
$$c = \frac{a_{13}(a_{11} - a_{12}) + a_{11}a_{44}}{a_{11}a_{33} - a_{13}^2}, \qquad d = \frac{a_{11}^2 - a_{12}^2}{a_{11}a_{33} - a_{13}^2}.$$

Formulas (1)–(3) describe the stress–strain state at an arbitrary point of the transversely isotropic body of revolution. The stress function φ should satisfy the following equation:

$$\Delta_1\Delta_2\varphi \equiv \frac{\partial^4\varphi}{\partial r^4} + \frac{2}{r}\,\frac{\partial^3\varphi}{\partial r^3} - \frac{1}{r^2}\,\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^3}\,\frac{\partial\varphi}{\partial r} + A\,\frac{\partial^4\varphi}{\partial r^2\,\partial z^2} + \frac{A}{r}\,\frac{\partial^3\varphi}{\partial r\,\partial z^2} + B\,\frac{\partial^4\varphi}{\partial z^4} = 0. \tag{4}$$

Here, we have

$$\Delta_i = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{s_i^2}\,\frac{\partial^2}{\partial z^2} \qquad (i = 1, 2); \tag{5}$$

$$s_1 = \sqrt{\Big(a + c + \sqrt{(a+c)^2 - 4d}\,\Big)/(2d)}\,, \qquad s_2 = \sqrt{\Big(a + c - \sqrt{(a+c)^2 - 4d}\,\Big)/(2d)}\,,$$

$$A = 1/s_1^2 + 1/s_2^2, \qquad B = 1/(s_1^2 s_2^2).$$

Let us consider the case with real values of s_1 and s_2.

Using the method of finding polynomial solutions for canonical equations of mathematical physics, which is based on expansion of the sought solutions with respect to eigen (basis) functions [3–5], we find the polynomial solutions of Eq. (4) with variable coefficients. The invariant solution of Eq. (4) is written as

$$\varphi(r,z) = r^\alpha J(\eta), \tag{6}$$

where $\eta = z/r$ and α is an arbitrary real number. Substituting Eq. (6) into Eq. (4) and taking into account that $r^{\alpha-6} \neq 0$, we obtain

$$(\eta^4 + A\eta^2 + B)J^{\rm IV} - (2\alpha - 5)(2\eta^3 + A\eta)J^{\rm III} + [\eta^2(6(\alpha-2)^2 - 1) + A(\alpha-2)^2]J^{\rm II}$$

$$- (2\alpha - 3)[2(\alpha-1)(\alpha-2) - 1]\,\eta J^{\rm I} + \alpha^2(\alpha-2)^2 = 0, \tag{7}$$

where $J^{\rm I}$, $J^{\rm II}$, $J^{\rm III}$, and $J^{\rm IV}$ are the derivatives of the corresponding order with respect to the variable η. The solution of Eq. (7) is sought in the form of a series:

$$J(\eta) = \sum_{k=0}^{\infty} c_k \eta^k. \tag{8}$$

Substituting Eq. (8) into Eq. (7) and equating the coefficients of the series at identical powers of η, we obtain the recurrent formula

$$c_{k+4} = -\frac{\Delta_k}{\Delta_0}\,\frac{1}{B}\,c_k - \frac{\Delta_{k+2}}{\Delta_1}\,\frac{A}{B}\,c_{k+2}, \tag{9}$$

where

$$\Delta_k = k(k-1)(k-2)(k - 4\alpha + 7) + k(k-1)[6(\alpha-2)^2 - 1] - k(2\alpha - 3)[2(\alpha-1)(\alpha-2) - 1] + \alpha^2(\alpha-2)^2,$$

$$\Delta_{k+2} = (\alpha - k - 2)^2, \quad \Delta_0 = (k+4)(k+3)(k+2)(k+1), \quad \Delta_1 = (k+4)(k+3).$$

Using Eq. (9), we can obtain the final form of the solution at $\alpha = k + 2$:

$$\varphi(r,z) = \sum_{\alpha=2}^{N} A_\alpha r^\alpha P_k^\alpha(\eta) = A_2 r^2(c_{02} + c_{22}\eta^2) + A_3 r^3(c_{13}\eta + c_{33}\eta^3)$$

$$+ A_4 r^4\Big[c_{04}\Big(1 - \frac{8}{3B}\,\eta^4\Big) + c_{24}\Big(\eta^2 - \frac{A}{3B}\,\eta^4\Big)\Big]$$

$$+ A_5 r^5\Big[c_{15}\Big(\eta - \frac{8}{15B}\,\eta^5\Big) + c_{35}\Big(\eta^3 - \frac{A}{5B}\,\eta^5\Big)\Big]$$

$$+ A_6 r^6\Big[c_{06}\Big(1 - \frac{24}{B}\,\eta^4 + \frac{16A}{5B^2}\,\eta^6\Big) + c_{26}\Big(\eta^2 - \frac{4A}{3B}\,\eta^4 + \frac{8}{45}\Big(\frac{A^2}{B^2} - \frac{1}{B}\Big)\eta^6\Big)\Big] + \ldots + A_N r^N P_k^N(\eta). \tag{10}$$

Here, $P_k^\alpha(\eta)$ are the polynomials that form the general solution of Eq. (7) for different values of α, $c_{0\alpha}$, $c_{2\alpha}$, $c_{1\alpha}$, and $c_{3\alpha}$ are the coefficients forming the initial basis, and A_α are arbitrary coefficients to be determined (the number of these coefficients depends on the chosen method for solving the boundary-value problem and on the estimate of accuracy of the approximate solution. The trivial solution $P_k^0(\eta) = A_0$ is not considered. Solutions presented in terms of transcendental functions are also ignored, because only polynomial solutions are determined.

A particular case of the above-given relations for a transversely isotropic body is the relations for an isotropic body.

Let us consider the presentation of the solution of the problem posed via two functions that should satisfy equations of a lower order. Let us introduce functions $\varphi_1(r,z)$ and $\varphi_2(r,z)$ satisfying the equations [6, 8]

$$\Delta_1\varphi_1 = 0, \qquad \Delta_2\varphi_2 = 0, \tag{11}$$

where Δ_i are the operators of Eq. (5). In this case, all equations of axisymmetric deformation are satisfied, and the expressions for displacements and stresses have the following form:

$$u_r = \frac{\partial}{\partial r}(\varphi_1 + \varphi_2), \qquad u_z = \frac{\partial}{\partial z}(k_1\varphi_1 + k_2\varphi_2),$$

$$\sigma_r = -\frac{A_{11} - A_{12}}{r}\frac{\partial}{\partial r}(\varphi_1 + \varphi_2) - A_{44}\frac{\partial^2}{\partial z^2}[(1 + k_1)\varphi_1 + (1 + k_2)\varphi_2],$$

$$\sigma_\theta = \frac{A_{11} - A_{12}}{r}\frac{\partial}{\partial r}(\varphi_1 + \varphi_2) + \frac{\partial^2}{\partial z^2}[(A_{13}k_1 - A_{12}\lambda_1^2)\varphi_1 + (A_{13}k_2 - A_{12}\lambda_2^2)\varphi_2], \tag{12}$$

$$\sigma_z = \frac{\partial^2}{\partial z^2}[(A_{33}k_1 - A_{13}\lambda_1^2)\varphi_1 + (A_{33}k_2 - A_{13}\lambda_2^2)\varphi_2],$$

$$\tau_{rz} = A_{44}\frac{\partial^2}{\partial z\,\partial r}[(1 + k_1)\varphi_1 + (1 + k_2)\varphi_2].$$

In these equations,

$$k_i = \frac{A_{11}\lambda_i^2 - A_{44}}{A_{13} + A_{44}} \quad (i = 1, 2), \quad A_{11} = \frac{a_{33}a_{11} - a_{13}^2}{(a_{11} - a_{12})e}, \quad A_{12} = \frac{a_{13}^2 - a_{33}a_{12}}{(a_{11} - a_{12})e},$$

$$A_{13} = -\frac{a_{13}}{e}, \quad A_{33} = -\frac{a_{11} + a_{12}}{e}, \quad A_{44} = \frac{1}{a_{44}}, \quad A_{11} - A_{12} = \frac{1}{a_{11} - a_{12}},$$

$$e = (a_{11} + a_{12})a_{33} - 2a_{13}^2, \qquad \lambda_1 = 1/s_1, \qquad \lambda_2 = 1/s_2.$$

The invariant solution of Eqs. (11) is written in the form

$$\varphi(r, z) = (a_1 r + b_1)^\alpha J(\eta), \qquad \eta = \frac{a_2 z + b_2}{a_1 r + b_1} \tag{13}$$

(α, a_1, b_1, a_2, and b_2 are arbitrary real numbers). Note that Eq. (13) can also be used to solve Eq. (4). Substituting Eq. (13) into Eq. (11) and taking into account that $(a_1 r + b_1)^{\alpha-2} \neq 0$, we obtain

$$(\eta^2 + D^2/s_i^2)J^{\mathrm{II}} - [2(\alpha - 1) + H]\eta J^{\mathrm{I}} + [\alpha(\alpha - 1) + H\alpha]J = 0, \tag{14}$$

where $D^2 = a_2^2/a_1^2$ and $H = 1/a_1$. The solution of Eq. (14) is sought in the form of series (8). Substituting Eq. (8) into Eq. (14) and equating the coefficients of the series at identical powers of η, we find the recurrent formula

$$c_{k+2} = \frac{-k(k - 1) + k[2(\alpha - 1) + H] - \alpha(\alpha - 1 + H)}{(k + 2)(k + 1)}\Big(\frac{s_i}{D}\Big)^2 c_k, \tag{15}$$

which yields the final form of the solution of Eqs. (11) at $\alpha = k$:

$$\varphi(r, z) = \sum_{\alpha=0}^{N} A_\alpha U_k^\alpha(r, z) = A_0 + A_1(a_2 z + b_2) + A_2 c_{02}\Big[(a_1 r + b_1)^2 - \frac{(H + 1)s_i^2}{D^2}(a_2 z + b_2)^2\Big]$$

$$+ A_3 c_{13}\Big[(a_1 r + b_1)^2(a_2 z + b_1) - \frac{(H + 1)s_i^2}{3D^2}(a_2 z + b_2)^3\Big]$$

$$+ A_4 c_{04}\Big[(a_1 r + b_1)^4 - \frac{2(H + 3)s_i^2}{D^2}(a_2 z + b_2)^2(a_1 r + b_1)^2 + \frac{(H + 1)(H + 3)s_i^4}{3D^4}(a_2 z + b_2)^4\Big]$$

$$+ A_5 c_{15}\Big[(a_1 r + b_1)^4(a_2 z + b_2) - \frac{(2H + 6)s_i^2}{3D^2}(a_2 z + b_2)^3(a_1 r + b_1)^2 + \frac{(H + 1)(2H + 6)s_i^4}{30D^4}(a_2 z + b_2)^5\Big]$$

$$+ A_6 c_{06}\Big[(a_1 r + b_1)^6 - \frac{3(H + 5)s_i^2}{D^2}(a_2 z + b_2)^2(a_1 r + b_1)^4$$

$$+ \frac{(H + 3)(H + 5)s_i^4}{D^4}(a_2 z + b_2)^4(a_1 z + b_1)^2 + \frac{(H + 1)(H + 3)(H + 5)s_i^6}{15D^6}(a_2 z + b_2)^6\Big] \ldots + A_N U^N(r, z). \tag{16}$$

In these equations, $U_k^\alpha(r,z) = (a_1 r + b_1)^\alpha P_k^\alpha(\eta)$ and $P_k^\alpha(\eta)$ are polynomials that form the general solution of Eq. (14) for different values of α.

For all N and A_α, Eqs. (10) and (16) identically satisfy Eqs. (4) and (11) in the domain Ω, but they do not satisfy the boundary conditions. The form of the polynomials $P^\alpha(\eta)$ in Eqs. (10) and (16) depends on the choice of the arbitrary parameters $c_{0\alpha}$ and $c_{1\alpha}$ and internal parameters a_1, b_1, a_2, and b_2 in Eq. (16). These parameters can be different for each polynomial $P^\alpha(\eta)$ and are chosen on the basis of certain physical, geometric, or mathematical considerations. For instance, these parameters can be chosen in a manner that the system of linear algebraic equations to which the initial problem is reduced (in the case of linear boundary conditions) is not ill-posed. For convenience, we can assume that $c_{0\alpha} = c_{1\alpha} = a_1 = a_2 = 1$ and $b_1 = b_2 = 0$.

Let us consider other solutions. These exists a method of constructing new solutions from some known solutions, which does not require finding the final transformations, but this method is applicable only for linear homogeneous equations. For such equations, solutions depending on parameters generate new solutions at differentiation with respect to these parameters [9].

To solve particular problems of axisymmetric deformation, in particular, deformation of a hollow cylinder made of a transversely isotropic material, it is possible to use stress functions of the form [7]

$$\varphi(r,z) = (B_{0m} z^m + B_{m2} r^2 z^{m-2} + \ldots + B_{mk} r^k z^{m-k} + \ldots) \ln r$$
$$+ C_{0m} z^m + C_{m2} r^2 z^{m-2} + \ldots + C_{mk} r^k z^{m-k} + \ldots, \tag{17}$$

where $m = 0, 1, 2, 3, \ldots$; $k = 0, 2, 4, 6, \ldots, m$ for even values of m; $k = 0, 2, 4, 6, \ldots, m-1$ for odd values of m. In the case with a hollow cylinder, the stress functions have the form

$$\varphi(r,z) = A_{0m} z^m + A_{m2} r^2 z^{m-2} + \ldots + A_{mk} r^k z^{m-k} + \ldots, \tag{18}$$

where $m = 0, 1, 2, 3, \ldots$; $k = 0, 2, 4, 6, \ldots, m$ for even values of m; $k = 0, 2, 4, 6, \ldots, m-1$ for odd values of m. In the isotropic case, these exist solutions of Eqs. (11) in the form

$$\varphi(r,z) = c_0\Big(\ln r + \frac{1}{2\cdot 1} + b_0\Big) z^n + c_1\Big(\ln r + \frac{1}{2\cdot 2} + b_1\Big) z^{n-2} r^2 + c_2\Big(\ln r + \frac{1}{2\cdot 3} + b_2\Big) z^{n-4} r^4 + \ldots, \tag{19}$$

where $b_0 = 1$ and $b_j = -\dfrac{1}{2}\sum_{t=1}^{j} \dfrac{2t+1}{t(t+1)}$.

By virtue of linearity of the initial equations, the solutions of the problems can be constructed by combining Eqs. (16)–(19).

In the case of axisymmetric deformation of a body made of an isotropic material, the displacements of its points are expressed via two Timpe functions [7]

$$u_z = \frac{\partial\varphi}{\partial z} - \frac{1}{2(1-2\nu)}\,\frac{\partial\psi}{\partial z}, \qquad u_r = \frac{\partial\varphi}{\partial r} - \frac{1}{2(1-2\nu)}\,\frac{\partial\psi}{\partial r} + \frac{1-\nu}{1-2\nu}\,\frac{\psi}{r},$$

which should satisfy the following equations:

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 0, \qquad \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\,\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0. \tag{20}$$

The solution of the first equation of system (20) is a particular case of the solution of Eqs. (11). The solution of the second equation of system (20) is sought by the method considered above. Choosing $\psi = r^\alpha J(\eta)$ $(\eta = z/r)$, we obtain the recurrent formula $c_{k+2} = -\dfrac{(\alpha-k)(\alpha-k-2)}{(k+2)(k+1)}\,c_k$ for the reduced equation $(\eta^2+1)J^{\mathrm{II}} - (2\alpha-3)\eta J^{\mathrm{I}} + \alpha(\alpha-2)J = 0$, which allows us to obtain the final solution in the form

$$\psi(r,z) = \sum_{\alpha=0}^{N} A_\alpha U_k^\alpha(r,z) = A_0 + A_1 c_{11} z + A_2 c_{02} r^2 + A_3 c_{13} r^2 z + A_4 c_{04}(r^4 - 4r^2 z^2)$$
$$+ A_5 c_{15}\Big(r^4 z - \frac{4}{3} z^3 r^2\Big) + A_6 c_{06}(r^6 - 12 r^4 z^2 + 8 r^2 z^4) + \ldots + A_N U^N(r,z), \tag{21}$$

where $U_k^\alpha(r,z) = r^\alpha P_k^\alpha(\eta)$.

It should be noted that this solution coincides with the solution of the first equation of system (20), which was given in [7], at $s_i = a_1 = a_2 = 1$, $b_1 = b_2 = 0$, and an appropriate choice of the values of $c_{0\alpha}$ and $c_{1\alpha}$ in solution (16). In addition, owing to the choice of the parameters $c_{0\alpha}$ and $c_{1\alpha}$, solution (21) coincides with the solution of the second equation of system (20) presented in [7].

If the forces R_n and Z_n on the side surface are specified, the boundary conditions reduce to conditions on the meridian on the surface of revolution:

$$\sigma_r \cos(n, r) + \tau_{rz} \cos(n, z) = R_n, \qquad \tau_{rz} \cos(n, r) + \sigma_z \cos(n, z) = Z_n. \tag{22}$$

For the displacements $g_1(l)$ and $g_2(l)$ specified on the surfaces, we obtain the conditions

$$u_r = g_1(l), \qquad w = g_2(l). \tag{23}$$

In both cases, the specified forces and displacements are assumed to be functions of the arc l of the meridian of the radial section. The expansion coefficients A_α that ensure the best approximation of the boundary conditions (22) and (23) are found by the method of collocations or by the method of weighted residues [5].

Let us consider the next example. We calculate the stress state of a hollow cylinder under the action of a constant radial load on the side cylindrical surfaces (Lamé problem) [7]. The boundary conditions have the form

$$\sigma_r(a, z) = p_a, \qquad \sigma_r(b, z) = p_b, \qquad \tau_{rz}(a, z) = \tau_{rz}(b, z) = 0.$$

Here, a and b are the inner and outer radii of the cylinder; p_a and p_b are the normal radial loads on the inner and outer cylindrical surfaces, respectively. The solution of the problem is presented in the form

$$\varphi_0 = A_1\Big(\frac{z^2}{2} - \frac{r^2}{4}\Big) + A_3\Big(\ln r + \frac{1}{2}\Big), \qquad \psi_0 = A_2\,\frac{r^2}{2} - A_4,$$

whence it follows that

$$u_r = (A_2 - A_1)\,\frac{r}{2} + \frac{1}{r}\Big[A_3 - \frac{2(1-\nu)}{1-2\nu}\,A_4\Big], \qquad u_z = A_1 z.$$

Substituting these relations into Eqs. (2), we can obtain the problem solution in stresses:

$$\sigma_z = 2G\Big(A_1 + \frac{\nu}{1-2\nu}\,A_2\Big),$$

$$\sigma_\theta = 2G\Big\{-\frac{1}{2}\,A_1 + \frac{\nu}{2(1-2\nu)}\,A_2 + \frac{1}{r^2}\Big[A_3 - \frac{2(1-\nu)}{1-2\nu}\,A_4\Big]\Big\},$$

$$\sigma_r = 2G\Big\{-\frac{1}{2}\,A_1 + \frac{\nu}{2(1-2\nu)}\,A_2 - \frac{1}{r^2}\Big[A_3 - \frac{2(1-\nu)}{1-2\nu}\,A_4\Big]\Big\}, \qquad \tau_{rz} = 0.$$

In the case of a plane stress state ($\sigma_z = 0$), with allowance for the boundary conditions, we obtain

$$A_1 = \frac{\nu}{G(1+\nu)}\,\frac{p_a a^2 - p_b b^2}{b^2 - a^2}, \qquad \sigma_r = \frac{p_b b^2 - p_a a^2}{b^2 - a^2} - \frac{(p_b - p_a) b^2 a^2}{(b^2 - a^2) r^2},$$

$$A_2 = -\frac{1-2\nu}{G(1+\nu)}\,\frac{p_a a^2 - p_b b^2}{b^2 - a^2}, \qquad \sigma_\theta = \frac{p_b b^2 - p_a a^2}{b^2 - a^2} + \frac{(p_b - p_a) b^2 a^2}{(b^2 - a^2) r^2}.$$

Let us consider another example. We calculate the stress state of a hollow cylinder under the action of a radial load on the inner and outer side surfaces, which varies linearly over the axis [7]. The boundary conditions have the form

$$\sigma_r(a, z) = p_a z, \qquad \sigma_r(b, z) = p_b z, \qquad \tau_{rz}(a, z) = \tau_{rz}(b, z) = 0,$$

where $p_a = \text{const}$ and $p_b = \text{const}$. The solution is sought in the form

$$\varphi_1 = A_1\Big(\frac{z^3}{6} - \frac{r^2 z}{4}\Big) + A_3\Big(\ln r + \frac{1}{2}\Big), \qquad \psi_1 = 2A_2 r^2 z + A_4(-z).$$

Similar to the previous example, we find the displacements

$$u_r = -\frac{p_a}{E}\,\frac{a^2(1-\nu) r z}{b^2 - a^2}\Big(1 + \frac{1+\nu}{1-\nu}\,\frac{b^2}{r^2}\Big) + \frac{p_b}{E}\,\frac{b^2(1-\nu) r z}{b^2 - a^2}\Big(1 - \frac{1+\nu}{1-\nu}\,\frac{a^2}{r^2}\Big),$$

$$u_z = \frac{p_a}{E}\,\frac{a^2}{b^2 - a^2}\Big[\nu z^2 + \frac{(1-\nu)r^2}{2} + (1+\nu)b^2\Big(\ln\frac{r}{b} + \frac{1}{2}\Big)\Big] - \frac{p_b}{E}\,\frac{b^2}{b^2 - a^2}\Big[\nu z^2 + \frac{(1-\nu)r^2}{2} + (1+\nu)a^2\Big(\ln\frac{r}{b} + \frac{1}{2}\Big)\Big]$$

and the stresses

$$\sigma_\theta = -p_a z\,\frac{a^2}{b^2 - a^2}\Big(1 + \frac{b^2}{r^2}\Big) + p_b z\,\frac{b^2}{b^2 - a^2}\Big(1 + \frac{a^2}{r^2}\Big),$$

$$\sigma_r = -p_a z\,\frac{a^2}{b^2 - a^2}\Big(1 - \frac{b^2}{r^2}\Big) + p_b z\,\frac{b^2}{b^2 - a^2}\Big(1 - \frac{a^2}{r^2}\Big), \qquad \tau_{rz} = 0.$$

Let us consider a transversely isotropic body deformed by forces that induce deformation of the general type depending on three coordinates [10]. We introduce the Cartesian coordinates with the axis being directed normal to the plane of isotropy. In this case, we have

$$u = -\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial^2\varphi}{\partial y\,\partial z} + \frac{\partial\psi}{\partial x}, \quad w = \alpha\Big(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \gamma\,\frac{\partial^2\varphi}{\partial z^2}\Big).$$

Here $\alpha = A_{11}/(A_{13} + A_{44})$; $\gamma = A_{44}/(A_{13} + A_{44})$; A_{ij} are elastic constants [6].

The functions φ and ψ satisfy the equations

$$\Delta_1\Delta_2\varphi = 0, \qquad \Delta_0\psi = 0, \tag{24}$$

where $\Delta_i = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{1}{s_i^2}\,\dfrac{\partial^2}{\partial z^2}$ $(i = 0, 1, 2)$; $s_0^2 = A_{66}/A_{44}$; s_1 and s_2 are the parameters introduced earlier. The polynomial solutions of Eqs. (24) can be found in [3, 5].

The approach considered in the paper allows obtaining a fairly simple algorithm for consecutive construction of polynomial solutions of an arbitrary power α by the recurrent formulas (9), (15). The algorithm substantially decreases the dimension of the algebraic system of equations with respect to unknown coefficients, because the approximation is constructed only at the domain boundary. The results obtained can be used to solve applied problems of mechanics of deformable solids made of anisotropic materials, including problems of parametric identification and inverse problems.

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